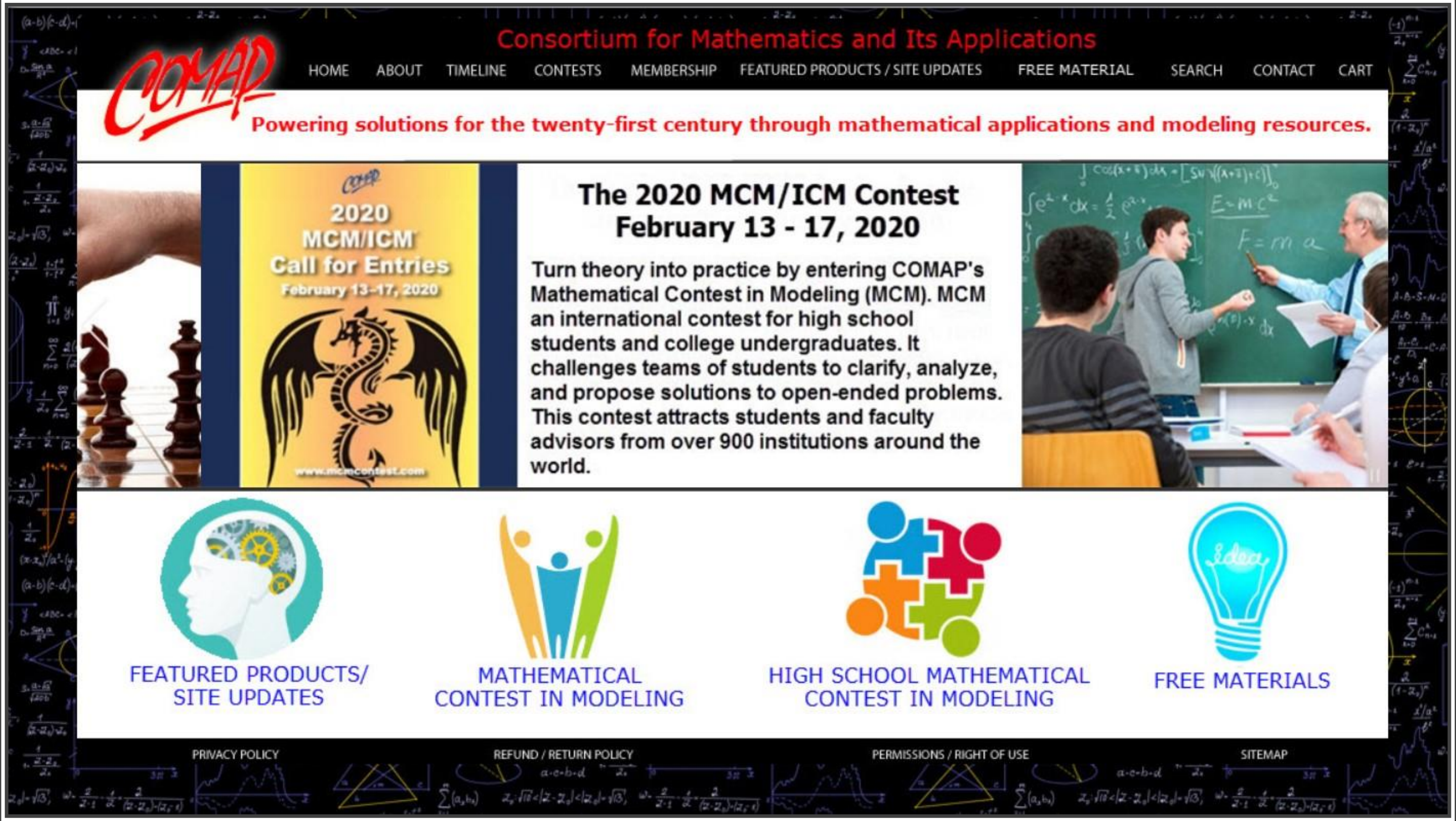
The Mathematics of the COMAP Modeling Contests

Problem A



Joint Mathematics Meeting

Denver, Colorado

January 16, 2020

Dr. Libby Schott
Florida SouthWestern State College

Who We Are | What We Do | How We Do It | Advisors | Contact

Who We Are

COMAP, the Consortium for Mathematics and Its Applications, is an award-winning non-profit organization whose mission is to improve mathematics education for students of all ages. Since 1980, COMAP has worked with teachers, students, and business people to create learning environments where mathematics is used to investigate and model real issues in our world.

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COMAP

Consortium for Mathematics and Its Applications

- COMAP develops curriculum materials . . .using mathematical tools to explore real-world problems.
- COMAP conducts research and analysis on the preparation of future mathematics educators . .
- COMAP provides technical assistance and professional development support to educators at all levels.
- COMAP has developed contest opportunities for students at both pre-college and post-secondary levels. . .
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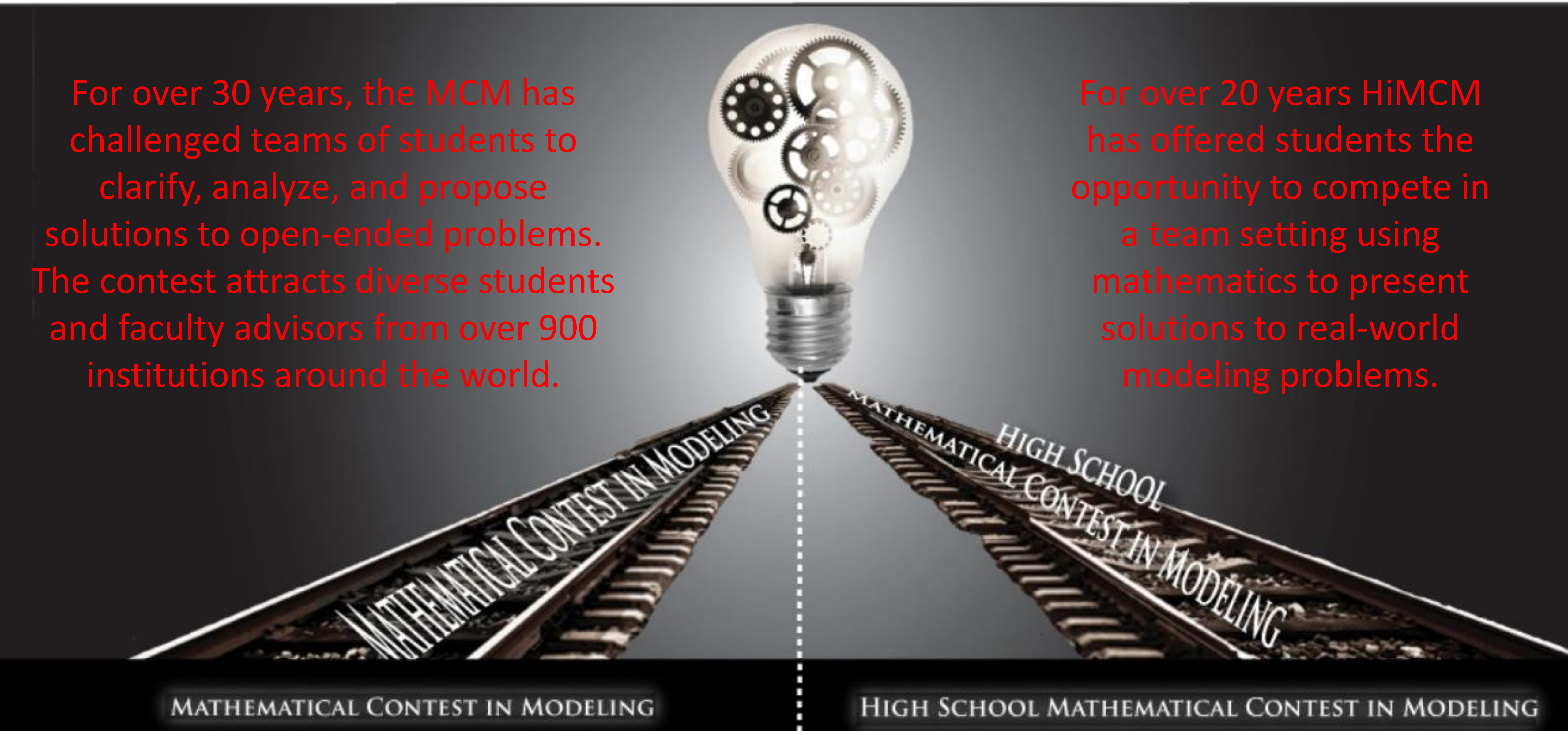
Powering solutions for the twenty-first century through mathematical applications and modeling resources.

For over 30 years, the MCM has challenged teams of students to clarify, analyze, and propose solutions to open-ended problems. The contest attracts diverse students and faculty advisors from over 900 institutions around the world.

For over 20 years HiMCM has offered students the opportunity to compete in a team setting using mathematics to present solutions to real-world modeling problems.



Some Previous Problem A Examples

1990: The Brain-Drug Problem

Researchers on brain disorders test the effects of the new medical drugs—for example, dopamine against Parkinson's disease—with intracerebral injections. To this end, they must estimate the size and the shape of the spatial distribution of the drug after the injection, in order to estimate accurately the region of the brain that the drug has affected.

The research data consist of the measurements of the amounts of drug in each of 50 cylindrical tissue samples (see **Figure 1** and **Table 1**). Each cylinder has length 0.76 mm and diameter 0.66 mm. The centers of the parallel cylinders lie on a grid with mesh 1 mm × 0.76 mm × 1 mm, so that the cylinders touch one another on their circular bases but not along their sides, as shown in the accompanying figure. The injection was made near the center of the cylinder with the highest scintillation count. Naturally, one expects that there is drug also between the cylinders and outside the region covered by the samples.

Estimate the distribution in the region affected by the drug.

One unit represents a scintillation count, or 4.753×10^{-13} mole of dopamine. For example, the table shows that the middle rear cylinder contains 28,353 units.

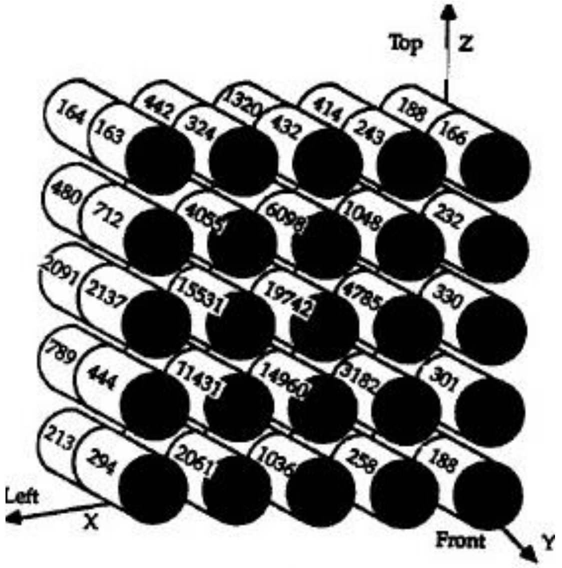


Figure 1. Orientation of the cylinders of tissue.

Table 1.

Amounts of drug in each of 50 cylindrical tissue samples.

Rear vertical section

164	442	1320	414	188
480	7022	14411	5158	352
2091	23027	28353	13138	681
789	21260	20921	11731	727
213	1303	3765	1715	453

Front vertical section

163	324	432	243	166
712	4055	6098	1048	232
2137	15531	19742	4785	330
444	11431	14960	3182	301
294	2061	1036	258	188

Other more recent problems

- 2015: **Eradicating Ebola**
- 2016: **A Hot Bath**
- 2017: **Managing The Zambezi River**
- 2018: **Multi-hop HF Radio Propagation**
- 2019: **Game of Ecology**

2019: Game of Ecology

In the fictional television series *Game of Thrones*, based on the series of epic fantasy novels *A Song of Ice and Fire*[1], three dragons are raised by Daenerys Targaryen, the "Mother of Dragons." When hatched, the dragons are small, roughly 10 kg, and after a year grow to roughly 30-40 kg. They continue to grow throughout their life depending on the conditions and amount of food available to them.

For the purposes of this problem, consider these three fictional dragons are living today. Assume that the basic biology of dragons described above is accurate. You will need to make some additional assumptions about dragons that might include, for example, that dragons are able to fly great distances, breath fire, and resist tremendous trauma. As you address the problem requirements, it should be clear how your assumptions are related to the physical constraints of the functions, size, diet, changes, or other characteristics associated with the animals.

Reference 1. Penguin Random House (2018). *A Song of Ice and Fire Series*. Retrieved from https://www.penguinrandomhouse.com/series/SOO/a-song-of-ice-and-fire/.

2019: Game of Ecology

Your team is assigned to analyze dragon characteristics, behavior, habits, diet, and interaction with their environment. To do so, you will have to consider many questions. At a minimum, address the following: What is the ecological impact and requirements of the dragons? What are the energy expenditures of the dragons, and what are their caloric intake requirements? How much area is required to support the three dragons? How large a community is necessary to support a dragon for varying levels of assistance that can be provided to the dragons? Be clear about what factors you are considering when addressing these questions.

As with other animals that migrate, dragons might travel to different regions of the world with very different climates. How important are the climate conditions to your analysis? For example, would moving a dragon between an arid region, a warm temperate region, and an arctic region make a big difference in the resources required to maintain and grow a dragon?

2019: Game of Ecology

Once your dragon analysis is complete, draft a two-page letter to the author of *A Song of Ice and Fire*, George R.R. Martin, to provide guidance about how to maintain the realistic ecological underpinning of the story, especially with respect to the movement of dragons from arid regions to temperate regions and to arctic regions.

While your dragon analysis does not directly apply to a real physical situation, the mathematical modeling itself makes use of many realistic features used in modeling a situation. Aside from the modeling activities themselves, describe and discuss a situation outside of the realm of fictional dragons that your modeling efforts might help inform and provide insight?

Modeling the Dragon’s Growth

- Logistics Model / von Bertalanffy growth function
- Piecewise Function based on size of the dragons from the show
- Differential Equation Model tying Dragon’s growth to vegetation availability and deer (or sheep) population
- Extrapolation based on data from Pterosaur and Komodo dragons

Modeling the Dragon's Energy Expenditure

Models Varied Based on Assumptions

- Maintaining Body Temperature
- Fire Breathing Capabilities
- Flight Requirements

Models Based On:

- The Gompertz curve (Time Series)
- Kleiber's Law
- Difference Equations
- Differential Equations
- Heat equation
- Lotka-Volterra predator prey equations

Modeling the Effects on the Ecosystem

- Analytic Hierarchy Process
- Simulations
- Varied Factors Relating to Dragon's Interaction with Ecosystems:
  - Availability of Food Sources
  - Various Types of Food Sources
  - Effects of Climate
  - Availability of Water
  - Energy Required to Migrate

Team 1917933:

Game of Ecology

Richard Montgomery High School, MD, USA

MAA Award & COMAP Scholarship Award

Team's Modeling Approach

Started by asking "How big is Balerion?"

- averaged a scaled-up pterosaur and a scaled-up lizard to approximate Balerion's weight.

Then used the von Bertalanffy Equation modified to model growth over time.

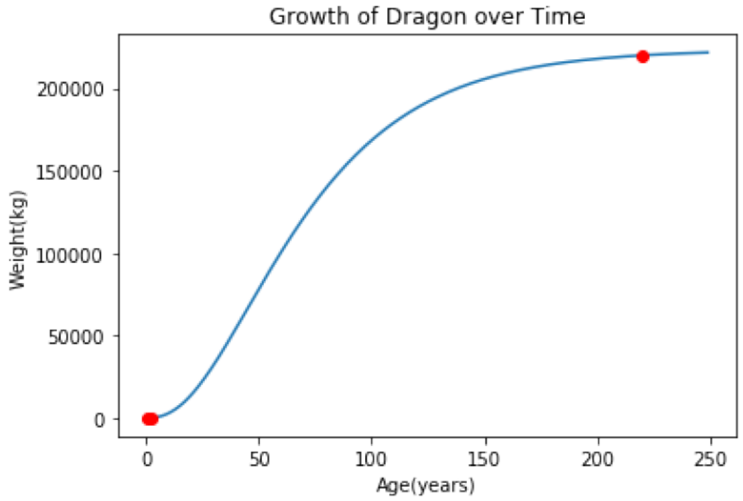
$$\mathrm{W}(t) = q\mathrm{L}^3$$

$$\mathrm{L}(t) = \mathrm{L}_\infty \left(1 - e^{-K(t+t_0)}\right)$$

thus becomes

$$\mathrm{W}(t) = \mathrm{W}_\infty \left(1 - e^{-K(t+t_0)}\right)^3$$

$$\mathrm{W}_\infty > \text{Balerion's weight}$$
$$t_0 = t \text{ value such that } \mathrm{W(t)} = 5$$
$$t = \text{years since birth}$$
$$\mathrm{K} = \text{growth rate}$$



$$\mathrm{W}(t) = 223348 \left(1 - e^{-.0238(x+1.203)}\right)^3$$

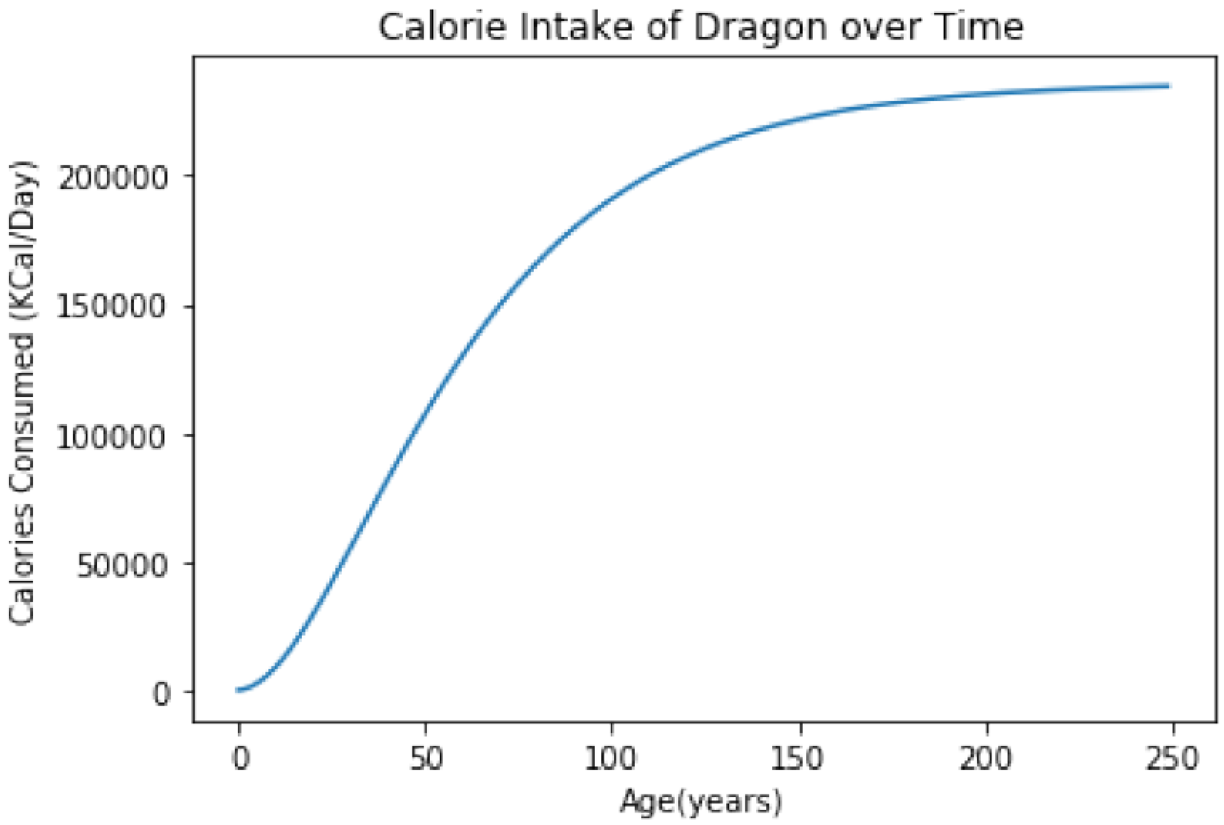
model for the basal metabolic rate of the dragons can be expressed as

$$B(t) = 20.65\left(\frac{M}{100}\right)^{-.25} = 65.3M^{-.25}$$

calories by day over time,

$$C(t) = 65.3(W(t))^{.75}$$

At Balerion's weight, 219,901 kg, we would expect him to need $65.3(219901)^{.75}$ kcal per day.



Activity Level	Multiplier
Sedentary	$\times 1.2$
Lightly Active	$\times 1.375$
Moderately Active	$\times 1.55$
Very Active	$\times 1.725$
Extra active	$\times 1.9$

A generalized equation can therefore be derived, as follows.

$$N = \frac{(W)(1490)}{A}$$

Where:

N is the number of calories per square mile that can be obtained in a given biome

W is the weight of meat consumed by the apex predator or a group of apex predators

A is the area of land, in square miles, required by the predator to fulfill their caloric requirements

Using this process for other biomes yields the following data, represented in the table below (sources can be found in the citations list).

Biome	Predator	Food Weight (lb)	Calories/Day	Area Required (square miles)	Calories/sq mi
Temperate Forest	Wolf	54	80460	50	1609.2
Tropical Rainforest	Tiger	17.5	26075	7.7	3386.363636
Boreal Forest	Lynx	2.5	3725	12.4	300.4032258
Desert	Coyote	2.5	3725	5	745
Savannah	Lion	262.5	391125	100	3911.25
Arctic Tundra	Wolf	54	80460	500	160.92

The upper bound of land requirement in different biomes is represented in the table below, once more using Balerion's weight.

Biome	Land Required per dragon (square miles)	Land Required for 3 Dragons (square miles)
Temperate Forest	710.8	2132.4
Tropical Rainforest	337.8	1013.4
Boreal Forest	3807.7	11423.2
Desert	1535.4	4606.1
Savannah	292.5	877.4
Arctic Tundra	7108.2	21324.7

	A	B	C	D	E	F	G
Substrates	Metabolic H_2O in g/kJ of oxidized substrate[a]	Energy in kJ/g substrate wet wt	Metabolic H_2O in g/g fuel (A × B)	H_2O content in g/g dry wt substrate[d]	Energy in kJ/g dry wt	H_2O content in g/kJ (D/E)	Total H_2O in g/kJ (A + F)
Fat	0.0272	39.6[e]	1.08	0.2	39.6	0.005	**0.032**
Protein	0.022	3.7[b]	0.08	3.65	18.4	0.198	**0.22**

Table taken from https://www.ncbi.nlm.nih.gov/pmc/articles/PMC5304341/ [16]

a: Values for calculating metabolic water production from indirect respiration calorimetry

b: Assuming that lean tissue contains approximately 73% water

c: Assuming that 2/3 of glycogen wet weight is water.

d: Water associated with electrolytes and hydration shells.

e: Assuming that fatty acids can be mobilized from adipose tissue without loss of tissue water.

However, we must also account for the food and land needed for the cows. The following formula can be used to model land needed [19]:

$$N = (A \times Y)/(0.04 \times W \times 365)$$

Where:

N is number of animals

A is of Acres

Y is yield per acre

W is average animal weight

Since the average weight of a cow is 2000 lbs and 8000 pounds of grass is grown per acre using alfafa, we can calculate the of acres needed, A, as:

$$5394x = (A \times 8000)/(0.04 \times 2000 \times 365)$$
$$= 19688x \text{ acres of land}$$

Team’s Conclusions

- Using Balerion, the largest dragon recorded, we determined that the largest dragons would need at least 3.6 cows a day.
- Based on their caloric intake, they would be able to produce this water solely from their metabolism for 7.5 hours of flight. We therefore recommend that in arid conditions, your dragons do not travel for longer than 7.5 hours without being provided with additional food or water
- We determined the necessary level of human intervention to provide dragons with their full caloric needs. When doing so, we discovered that a large-scale cow farm was needed, where cows were bred solely for the purpose of feeding the three dragons.

Team 1919022:

Dragons and Ecosystem: 1 + n Meta-population Model

Shanghai Jiao Tong University, China

Ben Fusaro Award

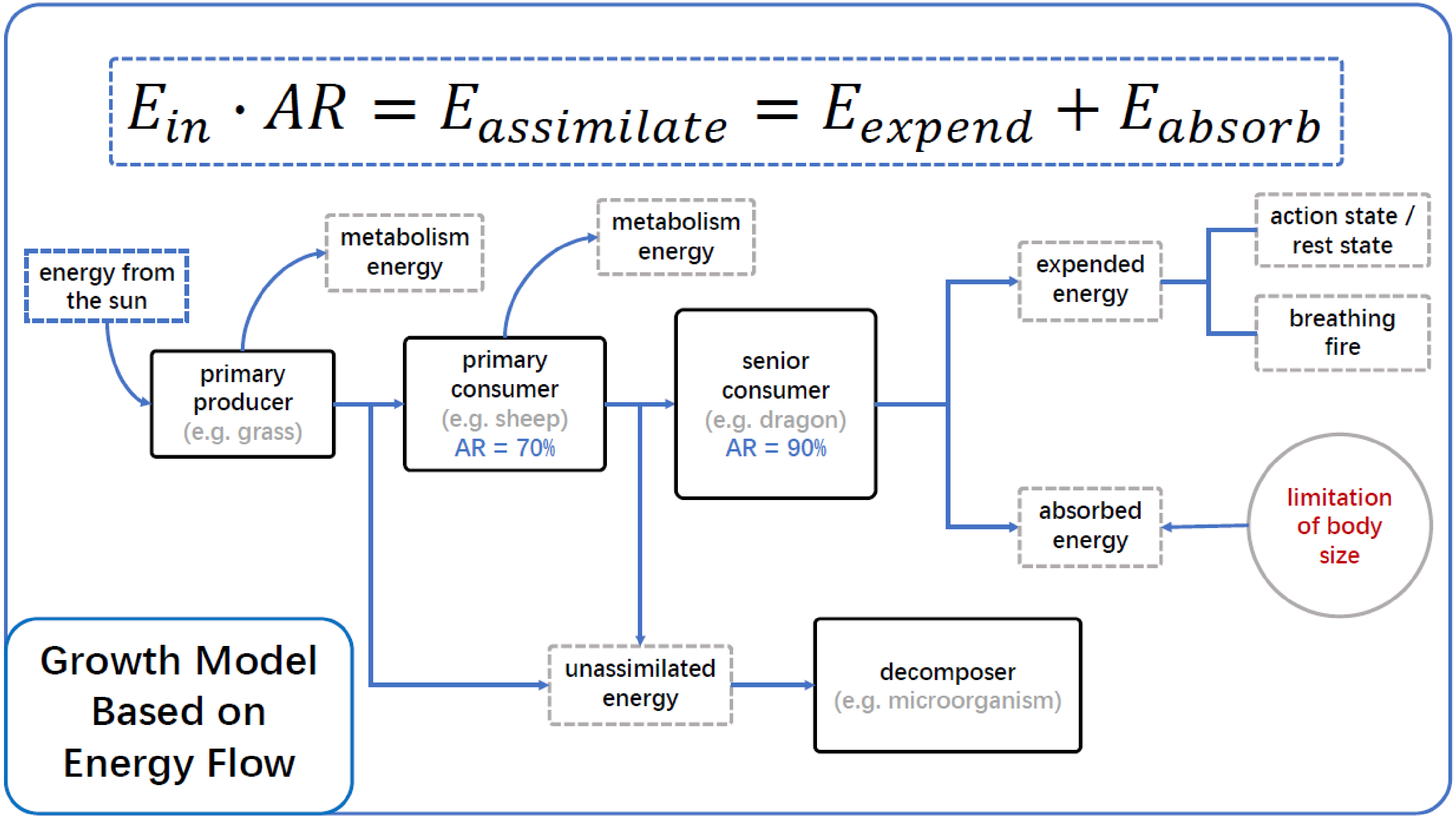


Figure 1: Framework of the Growth Model Based on Energy Flow

$$\frac{\mathrm{d}E_{expend}}{\mathrm{d}t} = \alpha_r q_{0r} M^{\frac{3}{4}} + \alpha_a q_{0a} M^{\frac{3}{4}},$$

$$E_{fire} = 6.69 \times 10^6 \times m$$

where m is the mass of the food the dargon eats.

$$AR = \frac{E_{assimilate}}{E_{in}},$$

$$\hat{q} = \frac{q}{M} = q_0 M^{b-1}.$$

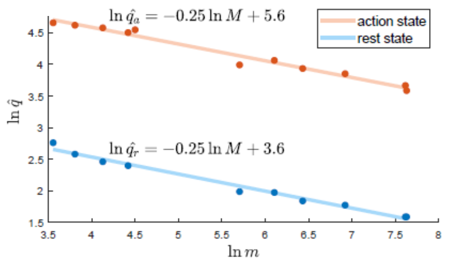


Figure 2: Linear fitting to calculate q_{0r} and q_{0a}

Table 2: Metabolic rate in rest state and action state

Flying animals	Body weight (g)	Metabolic rate ($\mathrm{cal} \cdot g^{-1} \cdot hr^{-1}$) Rest	Action
Budgerigar	35	15.8	105
Gull	300	7.3	54
pigeon	448	7.2	58
Gyrfalcon	2057	4.9	36

Used Keibler's Law for Basic Metabolic Rate

Considered Action and Rest

Determined Energy Expenditure

Determined Fire Energy

Determined Assimilation Rate

Determining the Growth Model

The energy a dragon assimilates is used in three aspects: absorption, motion expenditure and breathing fire. This process can be expressed as:

$$E_{assimilate} = E_{absorb} + E_{expend} + E_{fire}. \tag{5}$$

By estimation and calculation, E_{in} is:

$$E_{in} = 2 \times 10^7 \times m$$

$E_{assimilate}$ is:

$$\begin{aligned} E_{assimilate} - E_{fire} &= E_{in} \times AR - E_{fire} \\ &= E_{in} \times (AR - \frac{E_{fire}}{E_{in}}) \\ &\stackrel{def}{=} E_{in} \times K \end{aligned} \tag{6}$$

Using Equations. (4), (5) and (6), we can get Equation (7):

$$\frac{\mathrm{d}E_{in}}{\mathrm{d}t} \times K = \frac{\mathrm{d}E_{absorb}}{\mathrm{d}t} + (\alpha_r q_{0r} + \alpha_a q_{0a}) \cdot M_t^b \tag{7}$$

The dragon's absorption energy E_{absorb} is invested into growth, which is measured by the mass of the dragon in this paper. We define the dragon's mass at time point t by M_t, so the mass addition can be expressed as:

$$\lambda \frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\mathrm{d}E_{absorb}}{\mathrm{d}t} = \frac{\mathrm{d}E_{in}}{\mathrm{d}t} \times K - (\alpha_r q_{0r} + \alpha_a q_{0a}) \cdot M_t^b \tag{8}$$

The growth rates of living beings in nature aren't constant in different states of life. Von Bertalanffy [7] proposed the VB Equation to analyze the change of growth rate. It is reasonable to assume that dragons' growth also satisfies the VB Equation, so the relationship between the dragon's mass M and time t can be expressed as

$$\frac{\mathrm{d}M}{\mathrm{d}t} = 3kM^{2/3}(M_\infty^{1/3} - M^{1/3}), \tag{14}$$

where the unit of the time is *month*.

Integrating the Equation (14), we can get the equation 15:

$$M(t) = [(1 - e^{-kt})M_\infty^{1/3} + e^{-kt}M_0^{1/3}]^3. \tag{15}$$

The dragons are $10kg$ when hatched and are $30 \sim 40kg$ after a year grow. Using Equation (13), we can get

$$M(0) = 10kg, M(12) = 35kg, M(\infty) = 1.2 \times 10^5 kg.$$

So the k can be calculated:

$$k = 0.0020.$$

The growth process of the dragon is shown in the Figure. 3. The line means the relationship between the dragon's mass M and time t; the deeper is the color, the larger is the growth rate. So we can know that the dragon grows most fast aged between 30 and 80 years old.

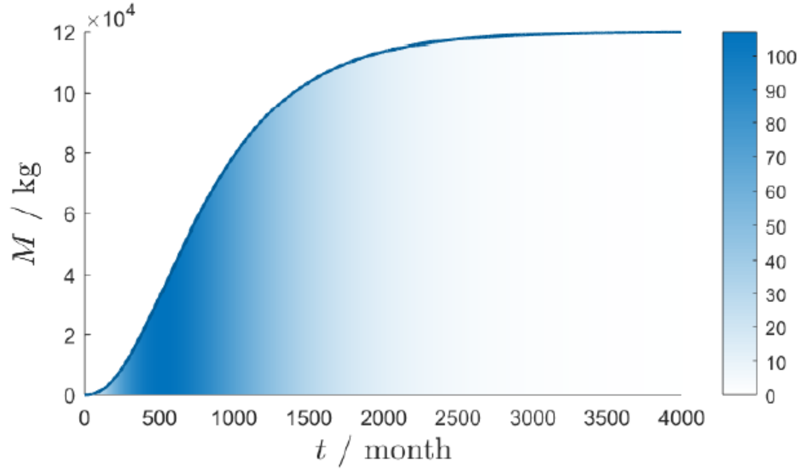


Figure 3: The relationship between the dragon's mass M and time t

Table 5: Net primary productivity per unit area in ecosystems

Type of ecosystem	Net primary productivity per unit area ($g \cdot m^{-2} \cdot a^{-1}$)		
	Area($10^6 km^2$)	Range	Average
Temperate deciduous forest	7.0	600∼2500	1200
Desert	18.0	10∼250	90
Ice sheet	24.0	0∼10	3

The loss and division of the habitat are the major reasons of the population's extinction. To investigate the ecosystem with n kinds of population, Tilman et al. [10] proposed the n-population Meta-population Model:

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = c_i p_i (1 - D - \Sigma_{j=1}^{i} p_j) - d_i p_i - \Sigma_{j=1}^{i-1} p_i c_i p_j \tag{17}$$

where $i = 1, 2, \cdots, n$, is the sort according to the population's competitive ability in the community; p_i is the habitat-occupancy proportion of population i; c_i is the migration rate of the population i; d_i is the mean death rate of the population i; D is the ratio of destroyed habitat to total habitat.

The minimal area to support a dragon's living is changing with time, so we set s as $s(t)$. It is reasonable that the scope of a dragon's activity is much larger than $s(t)$. In this paper, we set the dragon's living scope $S(t)$ as:

$$S(t) = 5s(t). \tag{18}$$

So the ratio D of destroyed habitat to total habitat can be expressed as:

$$D_1(t) = \frac{S(t)}{S_0}, \tag{19}$$

where S_0 is the area of the given region. So the Equation (17) can be written as:

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = c_i p_i (1 - \frac{S(t)}{S_0} - \Sigma_{j=1}^{i} p_j) - d_i p_i - \Sigma_{j=1}^{i-1} p_i c_i p_j \tag{20}$$

Human beings reconstruct the environment by repairing the habitat that dragons have destroyed. However, human beings cannot completely repair the destruction the dragons make. We set human beings' impact factor as μ. Besides, considering the observation, analysis and action of human beings have certain hysteresis, we set the time difference between dragons' destruction and human beings' reconstruction as ϵ. So the ratio of destroyed habitat to the total habitat D can be updated to Equation (21).

$$D_2(t) = \mu D_1(t - \epsilon) = \mu \frac{S(t - \epsilon)}{S_0} \tag{21}$$

Note that $D_2(t) = 0, t \leq \epsilon$. So Equation (22) can be expressed as:

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = c_i p_i (1 - \frac{S(t)}{S_0} + \mu \frac{S(t - \epsilon)}{S_0} - \Sigma_{j=1}^{i} p_j) - d_i p_i - \Sigma_{j=1}^{i-1} p_i c_i p_j \tag{22}$$

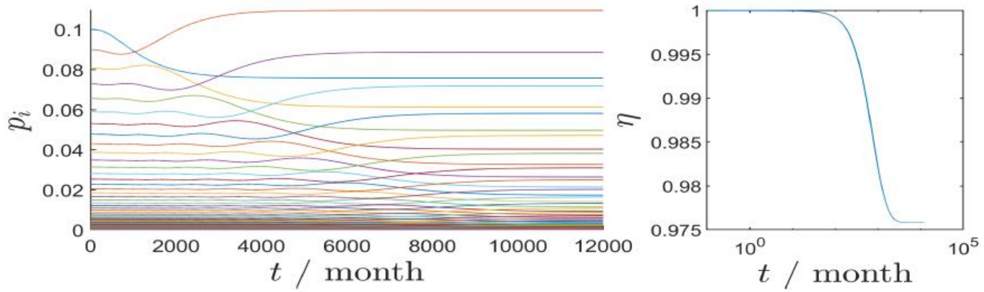


Figure 7: Simulation on the habitat-occupancy proportion of the population and the reduction ratio of the habitat-occupancy in temperate deciduous forest ($S_0 = 1 \times 10^9$). The reduction ratio of the population types $\xi(12000) = 100\%$.

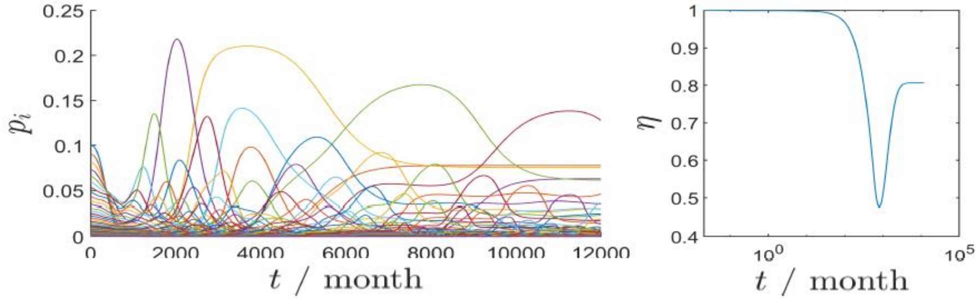


Figure 17: Simulation on the habitat-occupancy proportion of the population and the reduction ratio of the habitat-occupancy in temperate deciduous forest within human beings' effort ($S_0 = 2.5 \times 10^7$, $\mu = 0.8$, $\epsilon = 600$). The reduction ratio of the population types $\xi(12000) = 91.67\%$.

Team's Conclusions

To support three dragons, the area of the region must be equal to 2500 football fields

A small amount of human beings' help can save most of the populations

The more the human beings repair the habitat, the better and faster the ecosystem will recover.

Other Factors

- Sensitivity Analysis
- Model Strengths and Weaknesses
- Quality of the Letter to G.R.R.Martin
- Discussion of Applications to Other Situations

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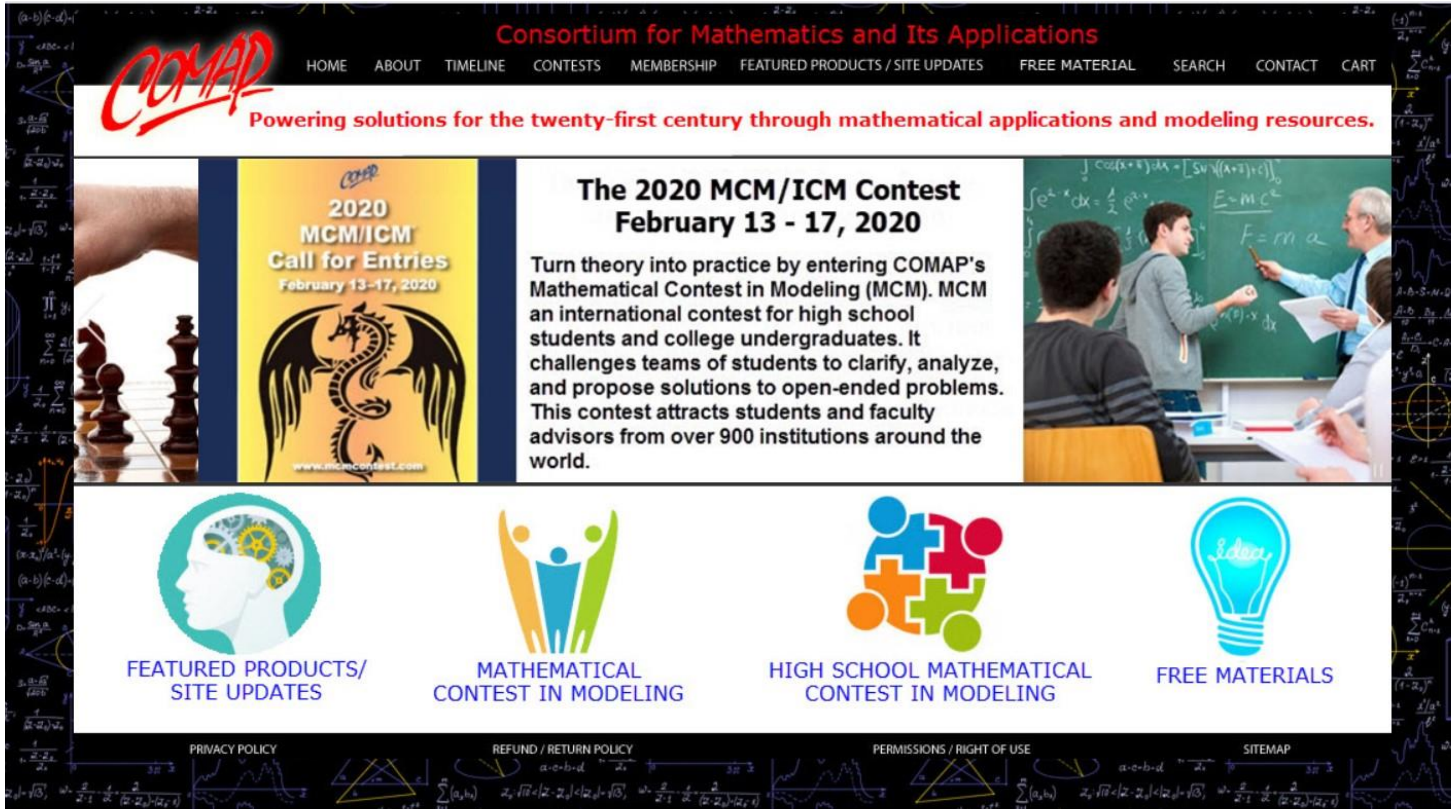

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Thank You!